Unit A9: System of Linear Equations in 2 or 3 Unknowns

Objective: (1) To solve a system of linear equations using Gaussian elimination.
(2) To recognize the existence and uniqueness of solution.

Detailed Content	Time Ratio	Notes on Teaching
9.1 Gaussian elimination and Echelon form	5	A matrix which satisfies the following 2 properties is said to be in Echelon form: (1) The 1st k rows are non-zero; the other rows are zero. (2) The 1st non-zero element in each non-zero row is 1, and it appears in a column to the right of the 1 st non-zero element of any preceding row. *Example:* The following 5 x 8 matrix is in Echelon form: $\begin{pmatrix} 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ Students should be able to solve a system of linear equations in two or three unknowns by using Gaussian elimination, which reduces a matrix in Echelon form by elementary operations on its rows. *Example:* Solve the system $\begin{cases} x_1 + 2x_2 + x_3 = 2 \\ 3x_1 + x_2 - 2x_3 = 1 \\ 4x_1 - 3x_2 - x_3 = 3 \\ 2x_1 + 4x_2 + 2x_3 = 4 \end{cases}$ The augmented matrix $\begin{pmatrix} 1 & 2 & 1 & 2 \\ 3 & 1 & -2 & 1 \\ 4 & -3 & -1 & 3 \\ 2 & 4 & 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -5 & -5 \\ 0 & -11 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\sim \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -11 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ The original system of equations is equivalent to the system of equations $\begin{cases} x_1 \quad - x_3 = 0 \\ \quad x_2 + x_3 = 1 \\ \quad\quad x_3 = 1 \end{cases}$ which gives $x_1 = 1$, $x_2 = 0$, $x_3 = 1$.

Detailed Content	Time Ratio	Notes on Teaching
9.2 Existence and uniqueness of solution	5	Students should be able to know the conditions for the existence and uniqueness of solution for a system of linear equations in two or three unknowns. For a system of linear equations in two unknowns: $a_1x + b_1y = d_1$ $a_2x + b_2y = d_2$ (i) If $\det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \neq 0$, the system has unique solution. Geometrically, the equations represent a pair of intersecting straight lines. (ii) If $\det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = 0$ and $\det\begin{pmatrix} d_1 & b_1 \\ d_2 & b_2 \end{pmatrix} \neq 0$, the system has no solution. Geometrically, the equations represent a pair of parallel (but not coincident) straight lines. (iii) If $\det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = 0$ and $\det\begin{pmatrix} d_1 & b_1 \\ d_2 & b_2 \end{pmatrix} = 0$, the system has infinite number of solutions. Geometrically, the equations represent a pair of coincident straight lines. For systems of equation in three unknowns, examples like the following should be mentioned. The corresponding geometrical meaning may be discussed if the students have grasped some ideas of three dimensional coordinate geometry. (i) In solving the equations $\begin{cases} 2x + y - z = 7 \\ 5x - 4y + 7z = 1 \\ 7x - 3y + 6z = 8 \end{cases}$, it is obvious that the third one is redundant. Teachers may discuss with the students on the method to obtain the solution $x = \frac{29 - 3\lambda}{13}, y = \frac{33 + 19\lambda}{13}, z = \lambda$ where λ is arbitrary. (ii) Solving equations like $\begin{cases} x + y + z = 3 \\ 2x - 3y + 2z = 1 \\ 3x - 2y + 3z = 7 \end{cases}$ which are inconsistent. Following this manner, the conditions for the existence and uniqueness of solution for a system of equations in three unknowns may be given in more abstract terms.
	10	